Homework 7 Solutions

Based on the third edition of *Abstract Algebra* of Dummit and Foote.

1 Section 4.5: Applications of the Sylow Theorems

Section 4.5, Exercise 14

Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

Solution. Factor

$$312 = 2^3 \cdot 3 \cdot 13.$$

Let n_{13} be the number of Sylow 13-subgroups. The Sylow theorems give

$$n_{13} \mid 24 \qquad \text{and} \qquad n_{13} \equiv 1 \pmod{13}.$$

Among the divisors of 24, only 1 is congruent to 1 modulo 13. Hence $n_{13} = 1$. The unique Sylow 13-subgroup is invariant under conjugation, so it is normal in G.

Section 4.5, Exercise 16

Let $|G| = pqr$, where p, q and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r.

Solution. Suppose, toward a contradiction, that neither a Sylow q-subgroup nor a Sylow r-subgroup is normal.

The Sylow conditions for r are

$$n_r \mid pq, \qquad n_r \equiv 1 \pmod{r}.$$

The proper divisors p and q of pq are both smaller than r, so neither can be congruent to 1 modulo r. Thus nonnormality forces $n_r = pq$. Distinct subgroups of order r intersect trivially, and therefore they contribute

$$pq(r-1)$$

distinct nonidentity elements.

Similarly, $n_q \mid pr$ and $n_q \equiv 1 \pmod{q}$. Since $n_q \neq 1$ and the divisor p is smaller than q, we must have $n_q \in \{r, pr\}$; in particular, $n_q \geq r$. Distinct subgroups of order q also intersect trivially, so the Sylow q-subgroups contribute at least

$$r(q-1)$$

distinct nonidentity elements. No nonidentity element can belong to both a subgroup of order q and one of order r. Consequently G would contain at least

$$1 + pq(r-1) + r(q-1) = pqr + \big(r(q-1) - pq + 1\big)$$

elements. But $r > q > p$ implies $r(q-1) > q(q-1)$, while $q - 1 \geq p$ gives $q(q-1) \geq pq$. Thus $r(q-1) > pq$, so the displayed total is greater than $pqr = |G|$, a contradiction. Hence a Sylow q-subgroup or a Sylow r-subgroup is normal.

Section 4.5, Exercise 22

Prove that if $|G| = 132$ then G is not simple.

Solution. Since $132 = 2^2 \cdot 3 \cdot 11$, the Sylow conditions give

$$n_{11} \mid 12, \qquad n_{11} \equiv 1 \pmod{11}.$$

Thus $n_{11} = 1$ or 12. If $n_{11} = 1$, the unique Sylow 11-subgroup is a nontrivial proper normal subgroup, and we are done.

Assume $n_{11} = 12$. Distinct subgroups of order 11 meet only in the identity, so they account for

$$12(11 - 1) = 120$$

nonidentity elements. Only 11 nonidentity elements remain.

For the Sylow 3-subgroups,

$$n_3 \mid 44, \qquad n_3 \equiv 1 \pmod{3},$$

so $n_3 \in \{1, 4, 22\}$. If $n_3 = 1$, again G has a nontrivial proper normal subgroup. The value $n_3 = 22$ is impossible, because it would require $22(3 - 1) = 44$ nonidentity elements outside the Sylow 11-subgroups. Thus, if no Sylow 3-subgroup is normal, then $n_3 = 4$, and these four subgroups account for $4(3 - 1) = 8$ further nonidentity elements.

Exactly three nonidentity elements are then left. Every Sylow 2-subgroup has order 4 and therefore consists of the identity together with three nonidentity elements. Hence every Sylow 2-subgroup must consist of the same four elements. There is only one Sylow 2-subgroup, so it is normal. In all cases G has a nontrivial proper normal subgroup; therefore G is not simple.

Section 4.5, Exercise 24

Prove that if G is a group of order 231 then $Z(G)$ contains a Sylow 11-subgroup of G and a Sylow 7-subgroup is normal in G.

Solution. We have $231 = 3 \cdot 7 \cdot 11$. For a Sylow 11-subgroup P,

$$n_{11} \mid 21, \qquad n_{11} \equiv 1 \pmod{11}.$$

The divisors of 21 are $1, 3, 7, 21$, and only 1 is congruent to 1 modulo 11. Thus P is unique and normal.

Likewise, for a Sylow 7-subgroup Q,

$$n_7 \mid 33, \qquad n_7 \equiv 1 \pmod{7}.$$

Among $1, 3, 11, 33$, only 1 has the required congruence. Hence Q is unique and normal.

It remains to prove that P is central. Conjugation on the normal subgroup P defines a homomorphism

$$G \longrightarrow \mathrm{Aut}(P).$$

Since $|P| = 11$, the group P is cyclic and $|\mathrm{Aut}(P)| = 10$. The order of the image divides both $|G| = 231$ and 10. Because $\gcd(231, 10) = 1$, the image is trivial. Thus every element of G centralizes P, and $P \leq Z(G)$.

Section 4.5, Exercise 26

Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal then G is abelian.

Solution. Let P be the normal Sylow 3-subgroup. Then $|P| = 3$, so P is cyclic. Conjugation induces a homomorphism

$$G \longrightarrow \mathrm{Aut}(P),$$

whose kernel is $C_G(P)$. Since P is abelian, $P \leq C_G(P)$, so the order of the image divides $[G : P] = 35$. It also divides $|\mathrm{Aut}(P)| = 2$. Hence the image is trivial, and therefore $P \leq Z(G)$.

Now $|G/P| = 35 = 5 \cdot 7$. In G/P, the number of Sylow 7-subgroups divides 5 and is congruent to 1 modulo 7, so it is 1. The number of Sylow 5-subgroups divides 7 and is congruent to 1 modulo 5, so it too is 1. These two normal Sylow subgroups commute: their commutator lies in their intersection, which is trivial. Their product is all of G/P, and each is cyclic of prime order. Thus

$$G/P \cong C_5 \times C_7 \cong C_{35}$$

is cyclic.

Choose $g \in G$ whose coset generates G/P. Every element of G has the form $g^a z$ with $z \in P$. Since $P \leq Z(G)$, any two such elements commute. Therefore G is abelian.

Section 4.5, Exercise 27

Let G be a group of order 315 which has a normal Sylow 3-subgroup. Prove that $Z(G)$ contains a Sylow 3-subgroup of G and deduce that G is abelian.

Solution. Let P be the normal Sylow 3-subgroup of G. Since $315 = 3^2 \cdot 5 \cdot 7$, we have $|P| = 9$. Every group of order p^2 is abelian, so P is abelian and hence $P \leq C_G(P)$.

Conjugation gives

$$G/C_G(P) \hookrightarrow \mathrm{Aut}(P).$$

The order of $G/C_G(P)$ divides $[G : P] = 35$. There are two possibilities for P. If $P \cong C_9$, then $|\mathrm{Aut}(P)| = \varphi(9) = 6$. If $P \cong C_3 \times C_3$, then

$$|\mathrm{Aut}(P)| = |\mathrm{GL}_2(\mathbb{F}_3)| = (3^2 - 1)(3^2 - 3) = 48.$$

Because $\gcd(35, 6) = \gcd(35, 48) = 1$, the conjugation image is trivial in either case. Thus $C_G(P) = G$, so $P \leq Z(G)$.

The quotient G/P has order 35. As in Exercise 4.5.26, both of its Sylow subgroups are unique, their product is direct, and

$$G/P \cong C_{35}$$

is cyclic. Since P is central and G/P is cyclic, every element of G can be written as $g^a z$ for a fixed $g \in G$ and some $z \in P$. Such elements commute with one another. Hence G is abelian.

Section 4.5, Exercise 33

Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

Solution. Because P is normal and is a Sylow p-subgroup, it is the unique Sylow p-subgroup of G: every Sylow p-subgroup is conjugate to P, and every conjugate of P equals P.

Let Q be any Sylow p-subgroup of H. Then Q is a p-subgroup of G, so it is contained in some Sylow p-subgroup of G. The latter must be P. Since also $Q \leq H$, we have

$$Q \leq P \cap H.$$

But $P \cap H$ is itself a p-subgroup of H. The maximality of the Sylow subgroup Q forces $Q = P \cap H$. Since this holds for every Sylow p-subgroup Q of H, the subgroup $P \cap H$ is the unique one.

Section 4.5, Exercise 44

Let p be the smallest prime dividing the order of the finite group G. If $P \in \mathrm{Syl}_p(G)$ and P is cyclic prove that $N_G(P) = C_G(P)$.

Solution. Write $|P| = p^a$. Conjugation by elements of $N_G(P)$ gives a homomorphism

$$N_G(P) \longrightarrow \mathrm{Aut}(P)$$

with kernel $C_G(P)$. Hence

$$N_G(P)/C_G(P) \hookrightarrow \mathrm{Aut}(P).$$

Because P is cyclic, P is abelian and therefore $P \leq C_G(P)$. Since P is a Sylow p-subgroup of G, the index $[N_G(P) : C_G(P)]$ is not divisible by p. On the other hand,

$$|\mathrm{Aut}(P)| = \varphi(p^a) = p^{a-1}(p-1),$$

so the embedding shows that $[N_G(P) : C_G(P)]$ divides $p^{a-1}(p-1)$. Its coprimality with p therefore implies

$$[N_G(P) : C_G(P)] \mid p - 1.$$

This index also divides $|G|$. If it were greater than 1, it would have a prime divisor smaller than p, which would then divide $|G|$, contradicting the minimality of p. Thus the index is 1, and $N_G(P) = C_G(P)$.

Section 4.5, Exercise 45

Find generators for a Sylow p-subgroup of S_{2p}, where p is an odd prime. Show that this is an abelian group of order p^2.

Solution. Define the disjoint p-cycles

$$\alpha = (1\ 2\ \cdots\ p), \qquad \beta = (p+1\ p+2\ \cdots\ 2p).$$

Because their supports are disjoint, α and β commute. Both have order p, and

$$\langle \alpha \rangle \cap \langle \beta \rangle = 1.$$

Consequently

$$P = \langle \alpha, \beta \rangle = \langle \alpha \rangle \times \langle \beta \rangle \cong C_p \times C_p.$$

Thus P is abelian and has order p^2.

Finally, since p is odd, the largest power of p dividing $|S_{2p}| = (2p)!$ is p^2. The subgroup P has exactly this order, so it is a Sylow p-subgroup of S_{2p}.

2 Section 4.6: The Simplicity of A_n

Section 4.6, Exercise 1

Prove that A_n does not have a proper subgroup of index $< n$ for all $n \geq 5$.

Solution. Fix $n \geq 5$ and suppose, for a contradiction, that $H < A_n$ has index

$$m = [A_n : H] < n.$$

The action of A_n by left multiplication on the m cosets of H produces a homomorphism

$$\varphi : A_n \longrightarrow S_m.$$

Its kernel is a normal subgroup of A_n. Moreover, $\ker \varphi \leq H$, because an element that fixes every coset fixes the coset H in particular. Since H is proper, the kernel cannot be all of A_n.

For $n \geq 5$, the group A_n is simple. Therefore $\ker \varphi = 1$, so φ is injective. It follows that

$$\frac{n!}{2} = |A_n| \leq |S_m| = m! \leq (n-1)!.$$

But $n!/2 > (n-1)!$ whenever $n > 2$, since their ratio is $n/2 > 1$. This contradiction proves that no proper subgroup of A_n can have index less than n.

Section 4.6, Exercise 3

Prove that A_n is the only proper subgroup of index $< n$ in S_n for all $n \geq 5$.

Solution. Fix $n \geq 5$, and let $H < S_n$ be a proper subgroup of index $m = [S_n : H] < n$. Set $K = H \cap A_n$. The map

$$A_n/K \longrightarrow S_n/H, \qquad aK \longmapsto aH,$$

is well defined and injective: if $aH = bH$ for $a, b \in A_n$, then $b^{-1}a \in H \cap A_n = K$. Hence

$$[A_n : K] \leq [S_n : H] = m < n.$$

By Exercise 4.6.1, A_n has no proper subgroup of index less than n. Therefore $K = A_n$, so $A_n \leq H$.

Finally,

$$2 = [S_n : A_n] = [S_n : H][H : A_n].$$

Since H is proper, $[S_n : H] \neq 1$. The displayed product therefore forces $[S_n : H] = 2$ and $[H : A_n] = 1$, so $H = A_n$. Conversely, $[S_n : A_n] = 2 < n$, completing the proof.